Additive Latin Transversals

using Combinatorial Nullstellensatz

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→ Discuss Latin transversals conjecture

→ Propose variant of Latin transversals conjecture

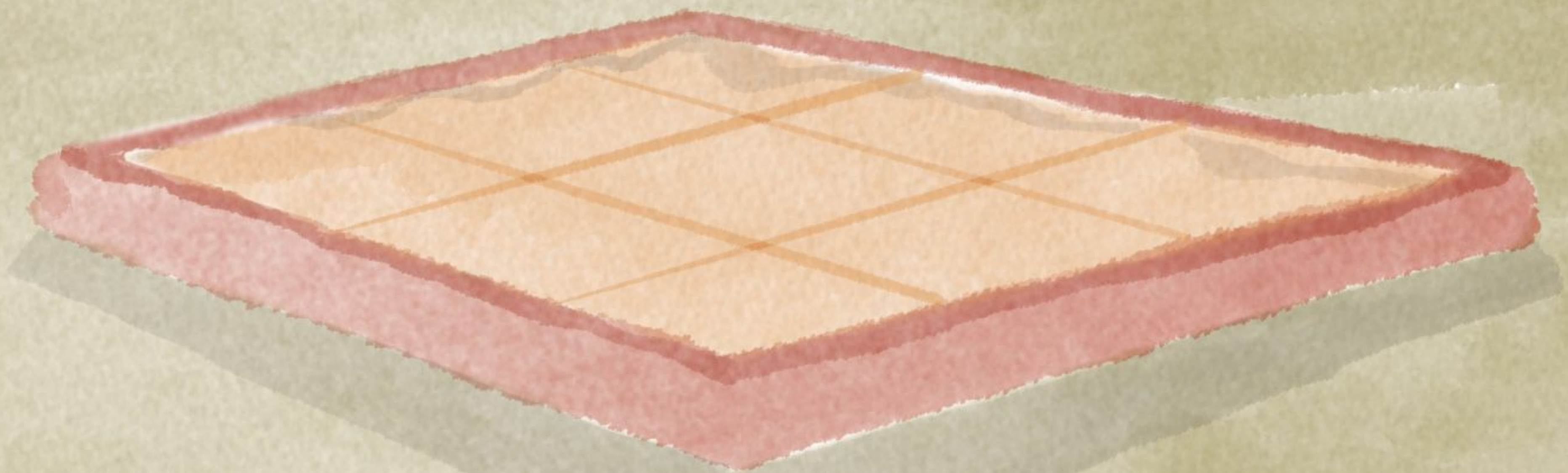
→ Reformulate as matching problem

→ Reformulate as Combinatorial Nullstellensatz problem

Latin Square

Latin Square

An arrangement of n numbers into an nxn grid such that each number appears exactly once in each row and column.



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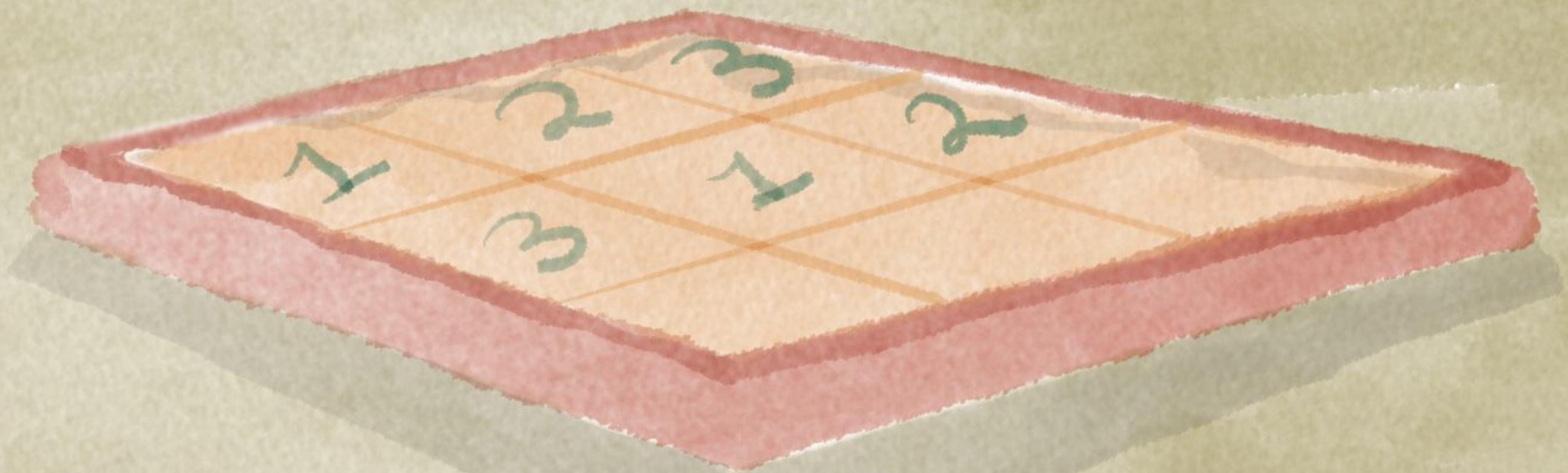
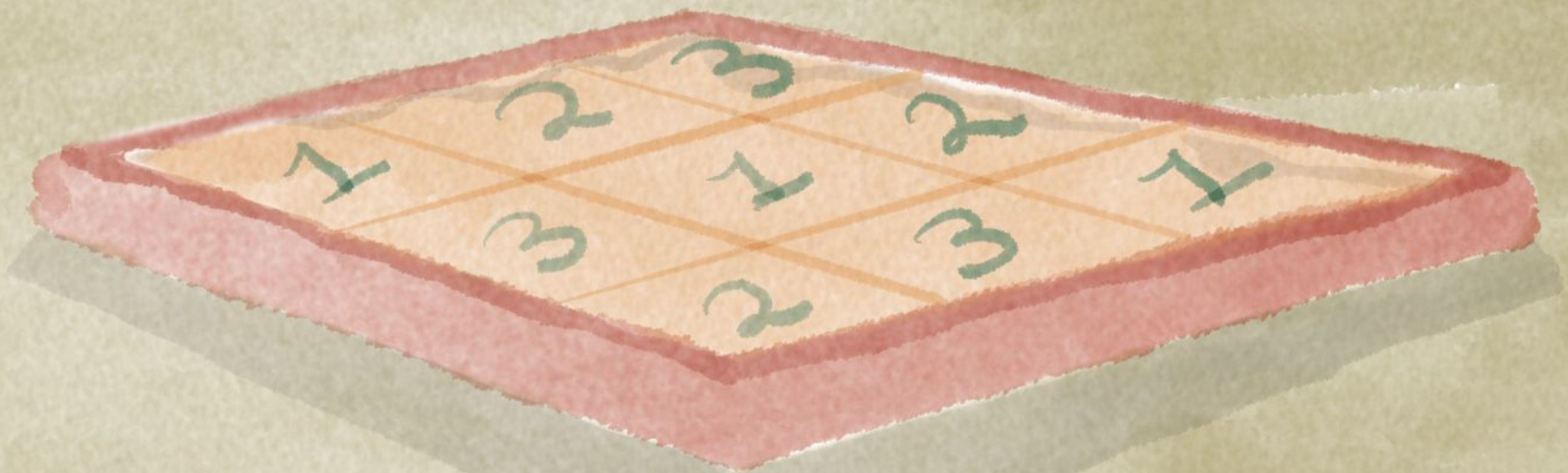


Latin Square

An arrangement of n numbers into an n×n grid such that each number appears <u>exactly once</u> in each row and column.

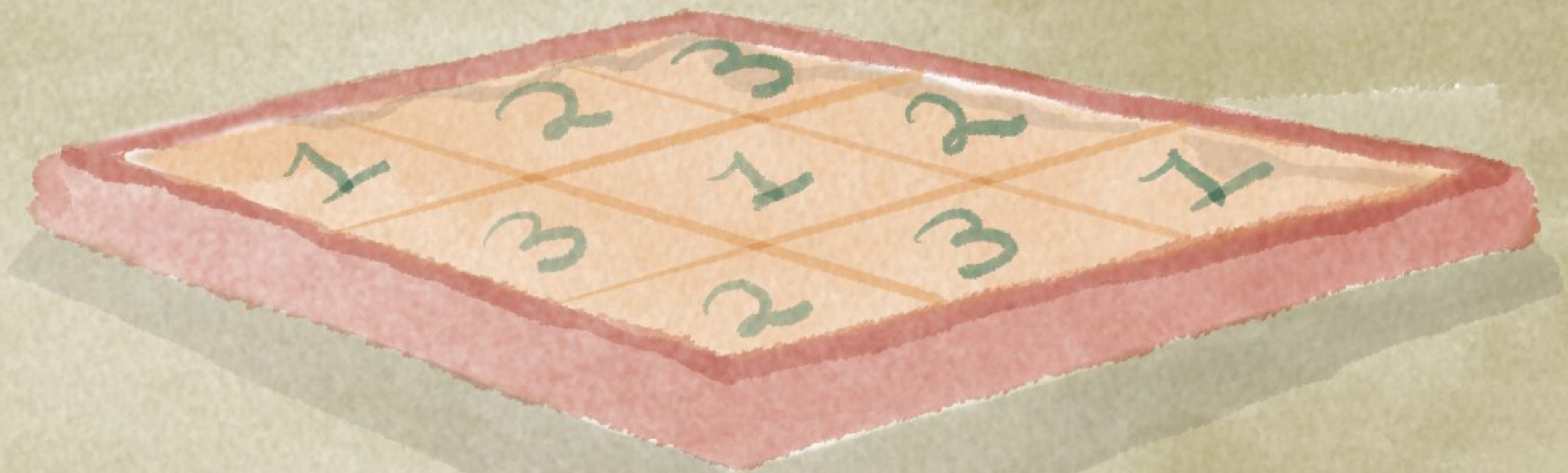


Latin Square

An arrangement of n numbers into an n×n grid such that each number appears exactly once in each row and column.

Latin Transversal

A set of n entries on a Latin square such that no two entries share the same row, column, or number.



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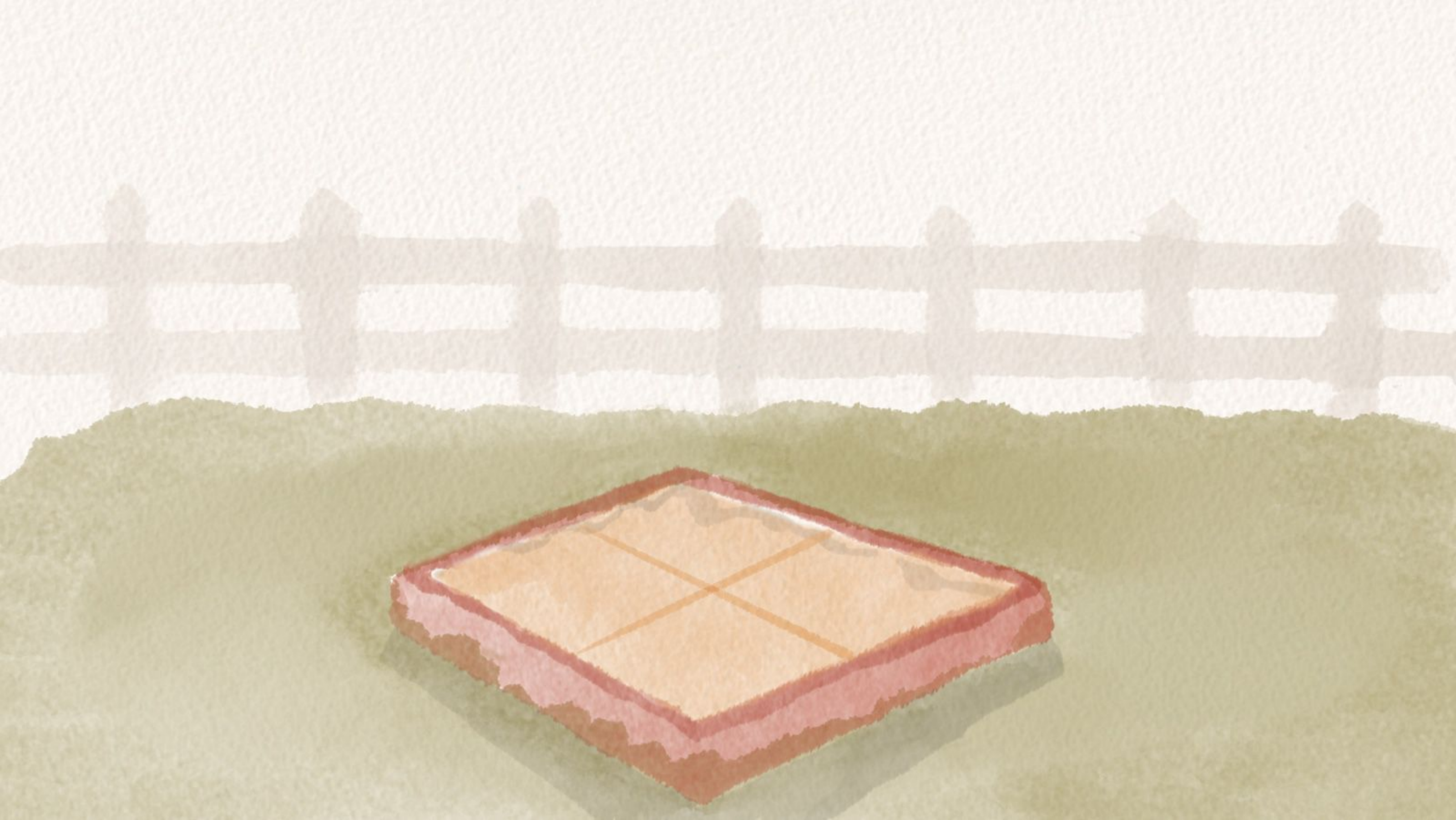
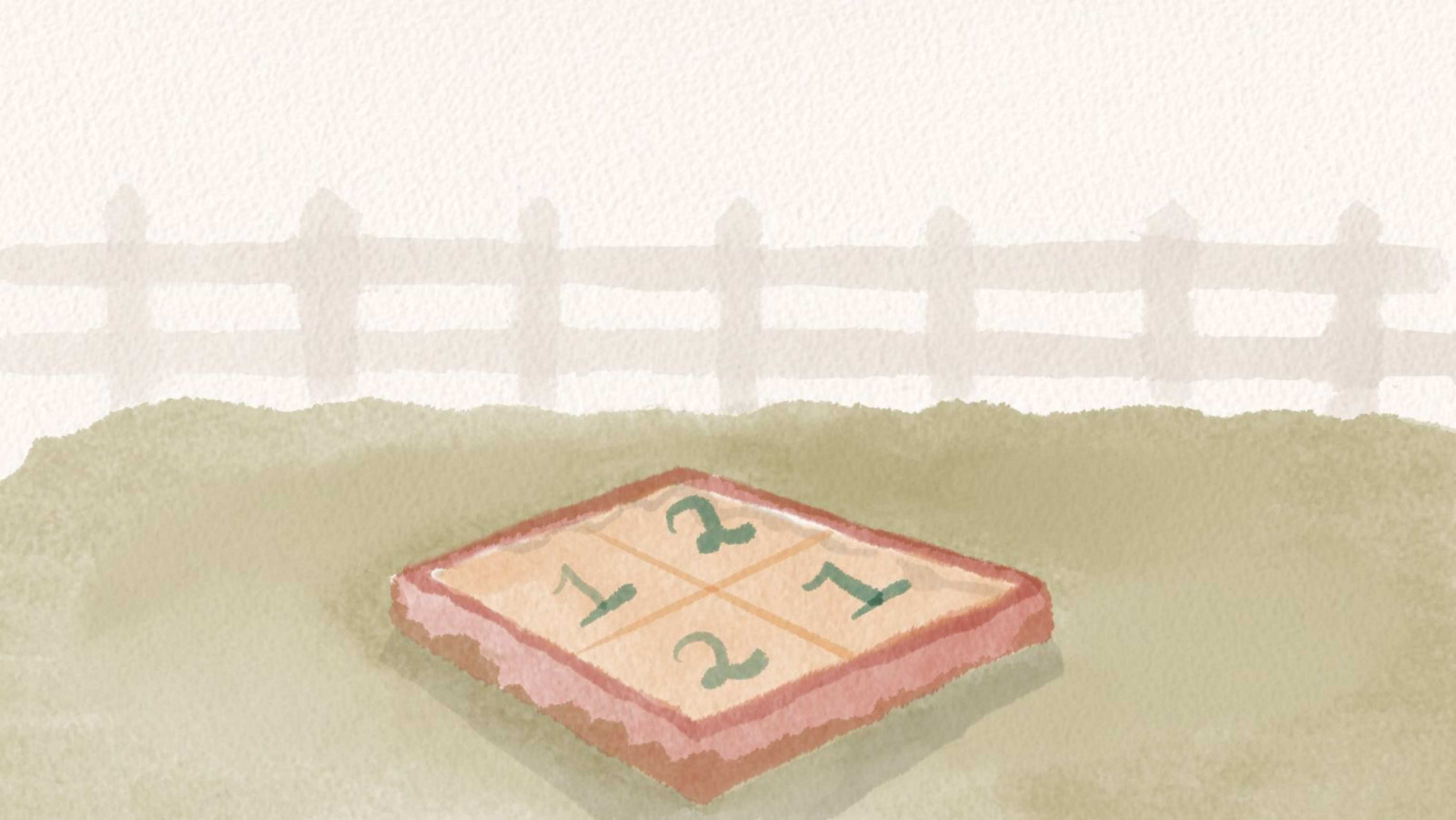



2
1
1
2

Theorem

When n is even, the existence of a Latin transversal is not guaranteed.

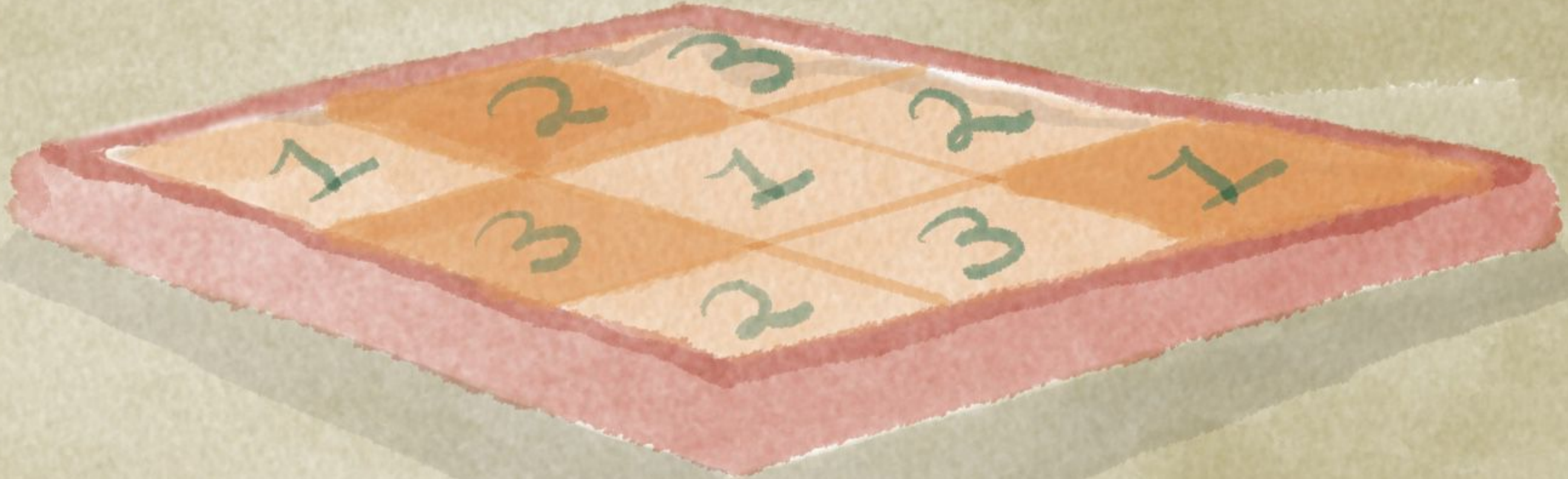

1 2 3
3 1 2
2 3 1

Theorem

When n is even, the existence of a Latin transversal is not guaranteed.

Conjecture

When n is odd, the existence of a Latin transversal is guaranteed.



Theorem

When n is even, the existence of a Latin transversal is not guaranteed.

Conjecture

When n is odd, the existence of a Latin transversal is guaranteed.

Let's examine a variant of this conjecture.

Additive Latin Transversals

using Combinatorial Nullstellensatz

→ Discuss Latin transversals conjecture

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→ Reformulate as matching problem

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Theorem

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We can find a Latin transversal...

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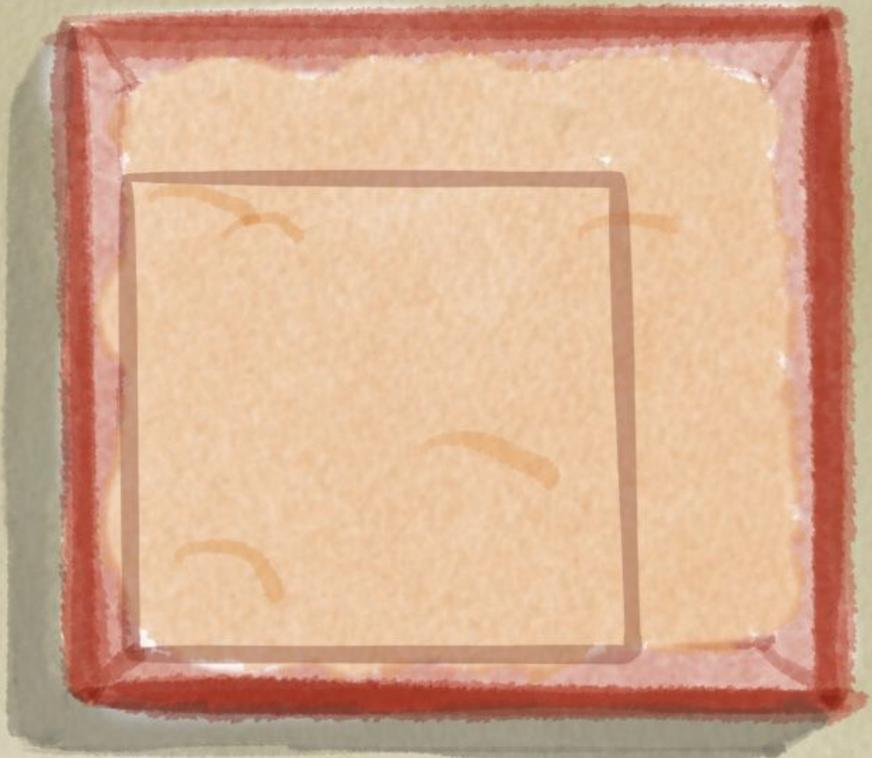
We can find a Latin transversal...

... on some subsquare of a Latin square



Theorem

We can find a Latin transversal...

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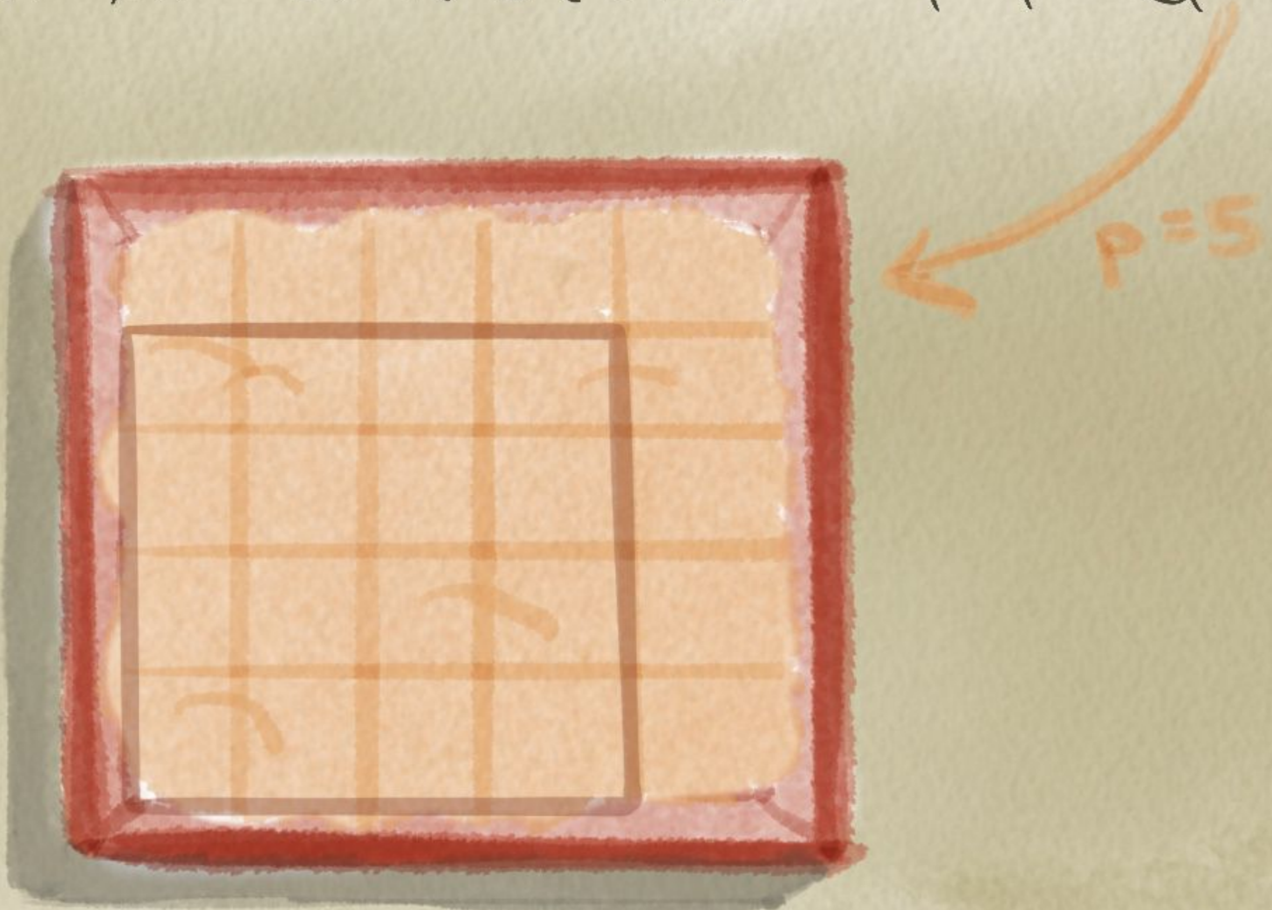
... when that Latin square is $p \times p$ (p is prime)



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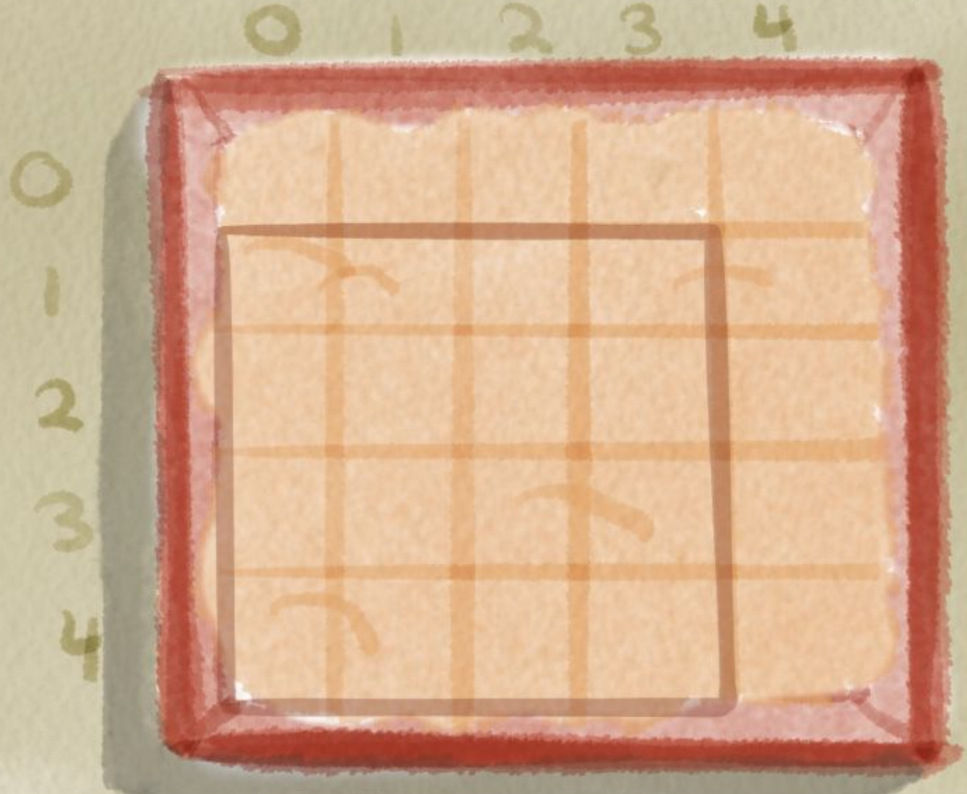

Theorem

We can find a Latin transversal...

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	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Theorem

We can find a Latin transversal...

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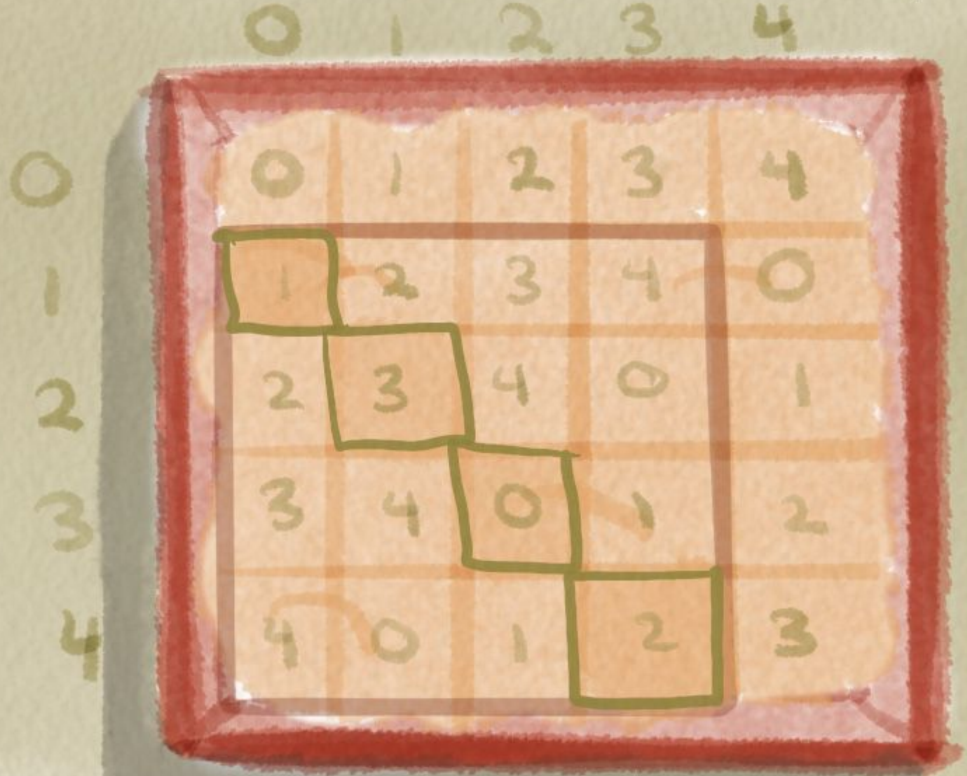

Additive Latin Transversals

using Combinatorial Nullstellensatz

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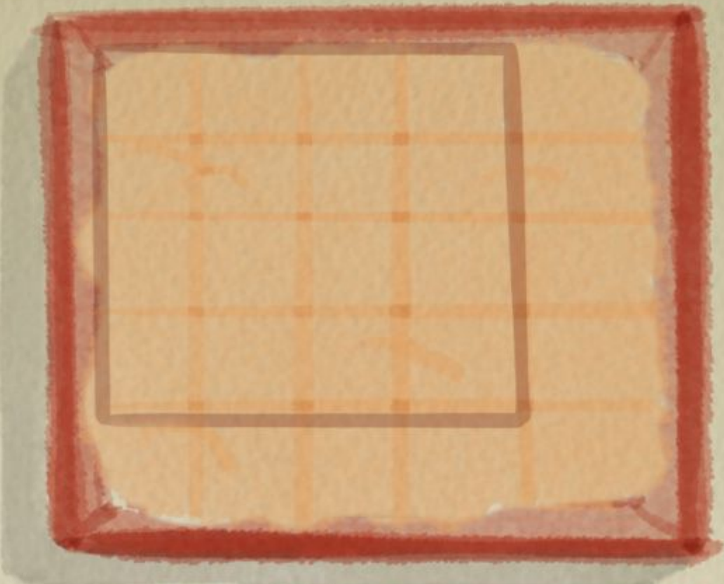
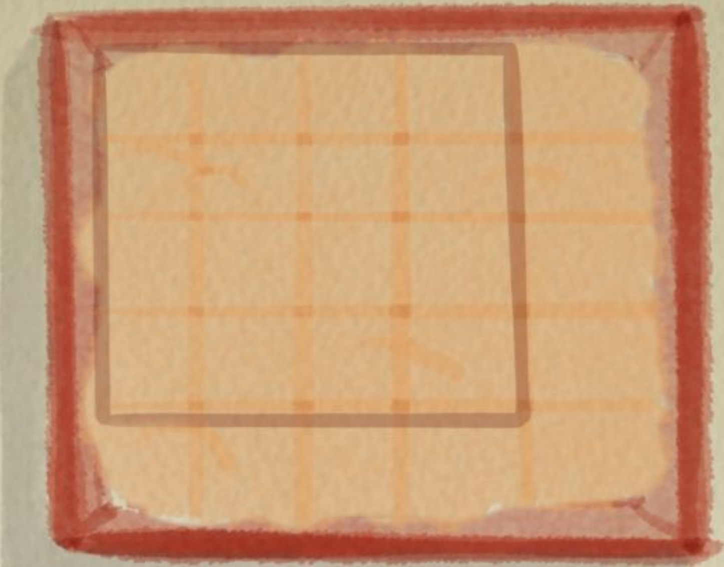
Theorem (Latin Squares)

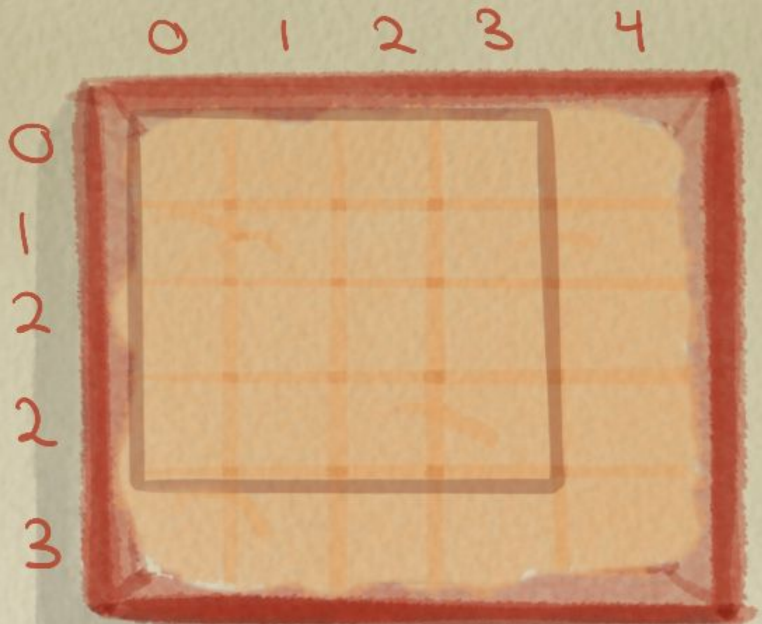


Theorem (Latin Squares) $\Longleftrightarrow$ Theorem (Matching)



Theorem (Latin Squares) $\iff$ Theorem (Matching)

Consider a subsquare on an addition table (mod p), of size $p \times p$, where p is prime. (Repetition in rows is allowed.)



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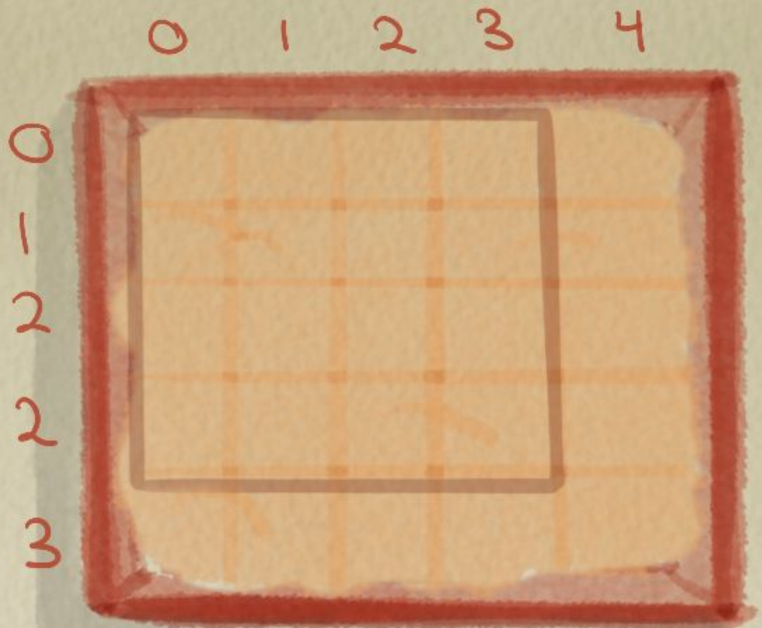
Consider a multiset A and set B, each of cardinality $k < p$, where p is prime.



A	B
0	0
1	1
2	2
2	3
2	

Theorem (Latin Squares) $\iff$ Theorem (Matching)

Consider a subsquare on an addition table (mod p), of size $p \times p$, where p is prime. (Repetition in rows is allowed.)

Then there exists a transversal s.t. no two cells share the same symbol.

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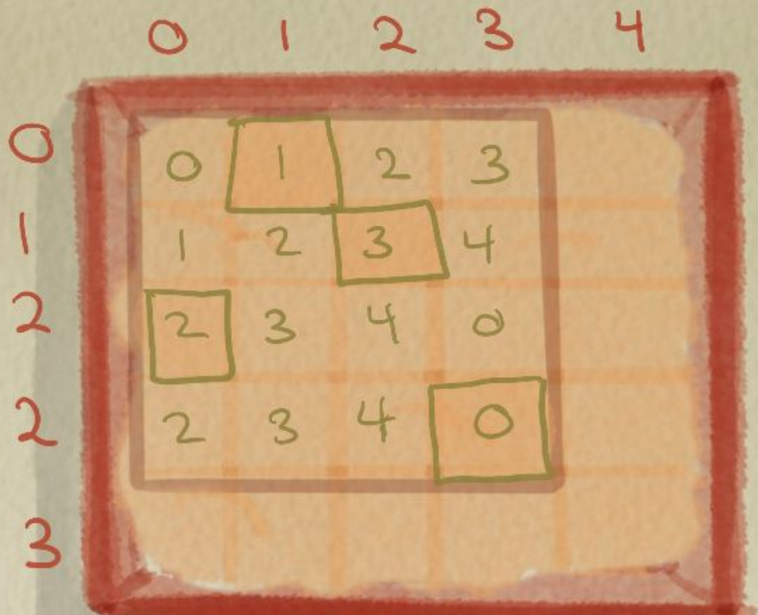

A	B
0	0
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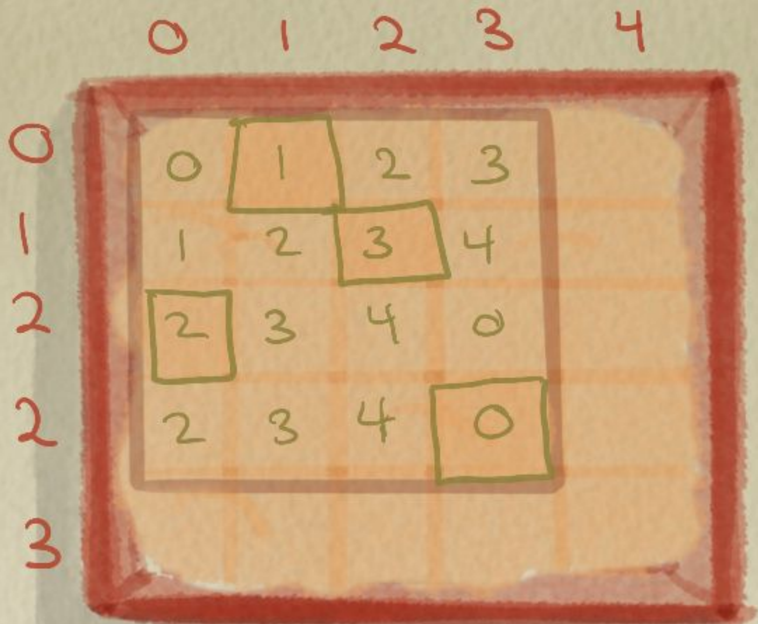
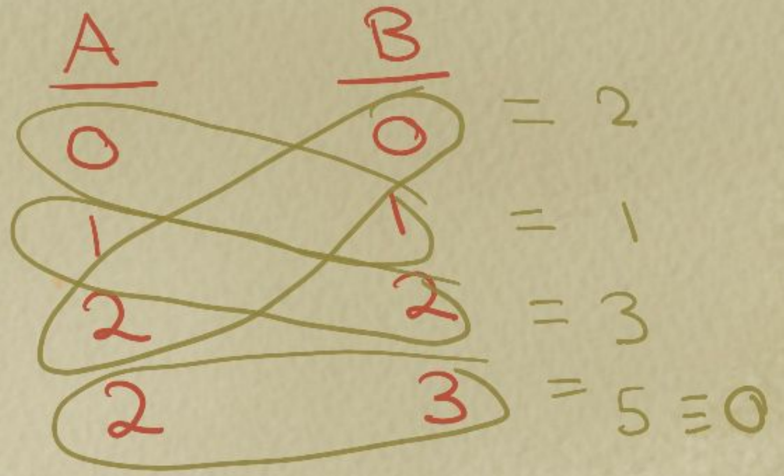
Theorem (Matching)

Consider a multiset A and set B, each of cardinality $k < p$, where p is prime.

Then there exists a numbering of elements $a_1 \dots a_k$ and $b_1 \dots b_k$ s.t. $a_i + b_i$ (mod p) are unique.





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0	$\boxed{0}$	1	2	3	
1	1	$\boxed{2}$	3	4	
2	2	3	$\boxed{4}$	0	
2	2	3	4	$\boxed{0}$	
3					

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A		B	
0	+	0	= 0
1	+	1	= 2
2	+	2	= 4
2	+	3	= 5

Theorem (Latin Squares) $\Longleftrightarrow$ Theorem (Matching)

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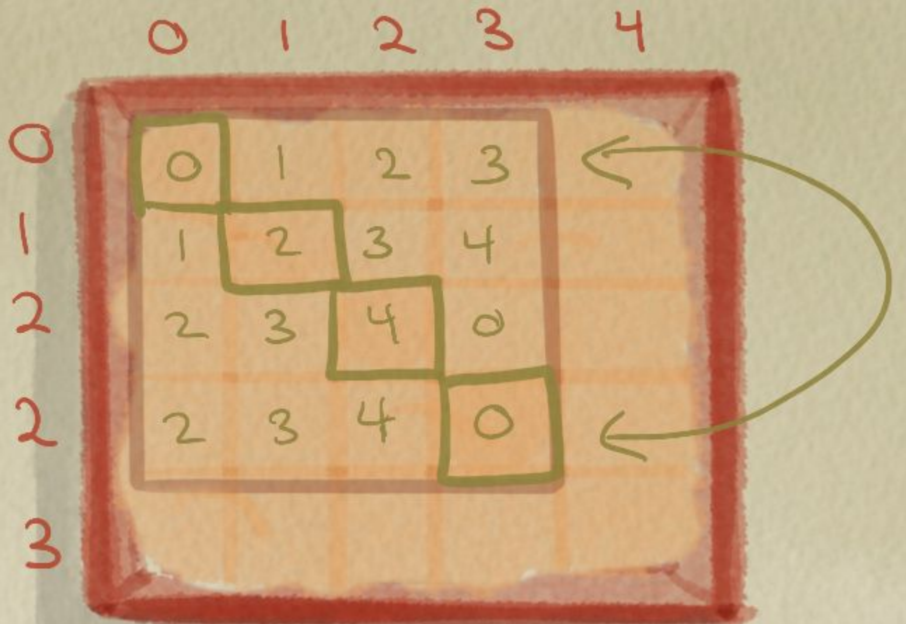

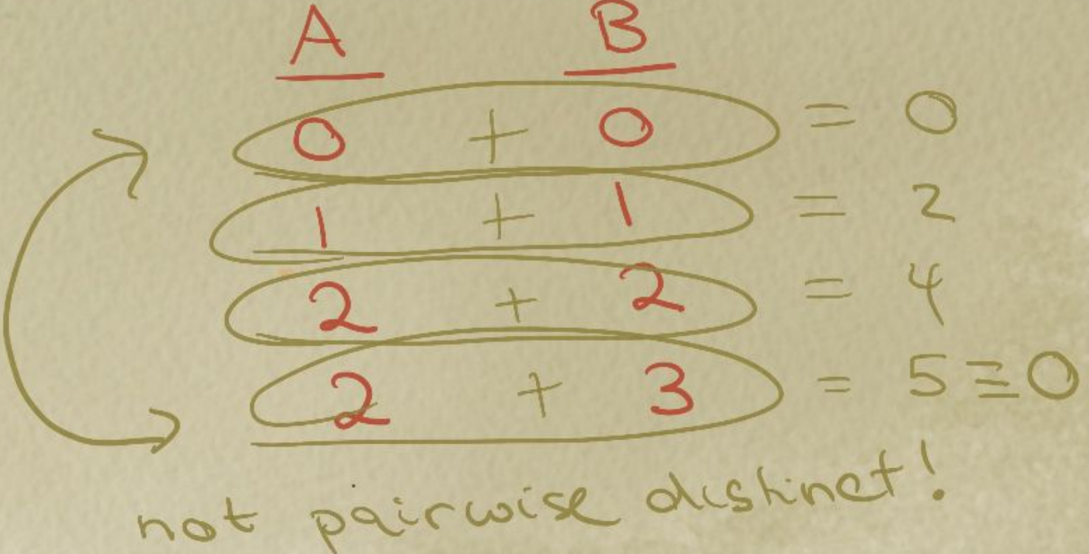

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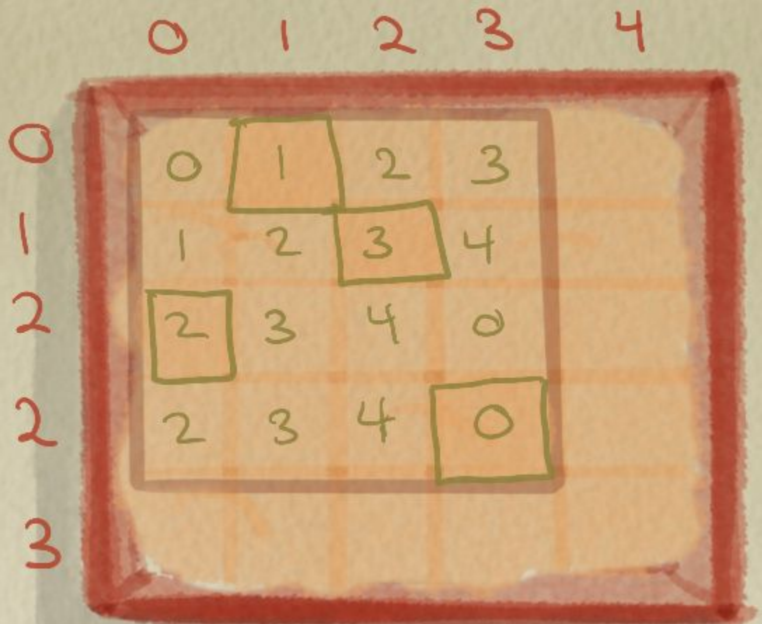

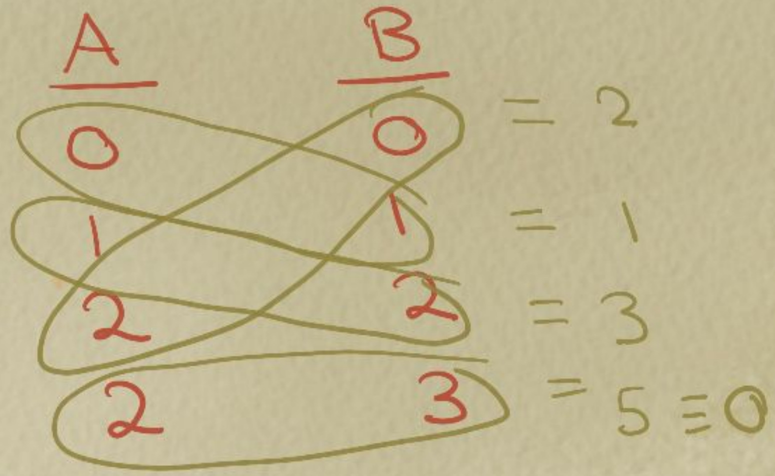

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$\longrightarrow$ Discuss Latin transversals conjecture

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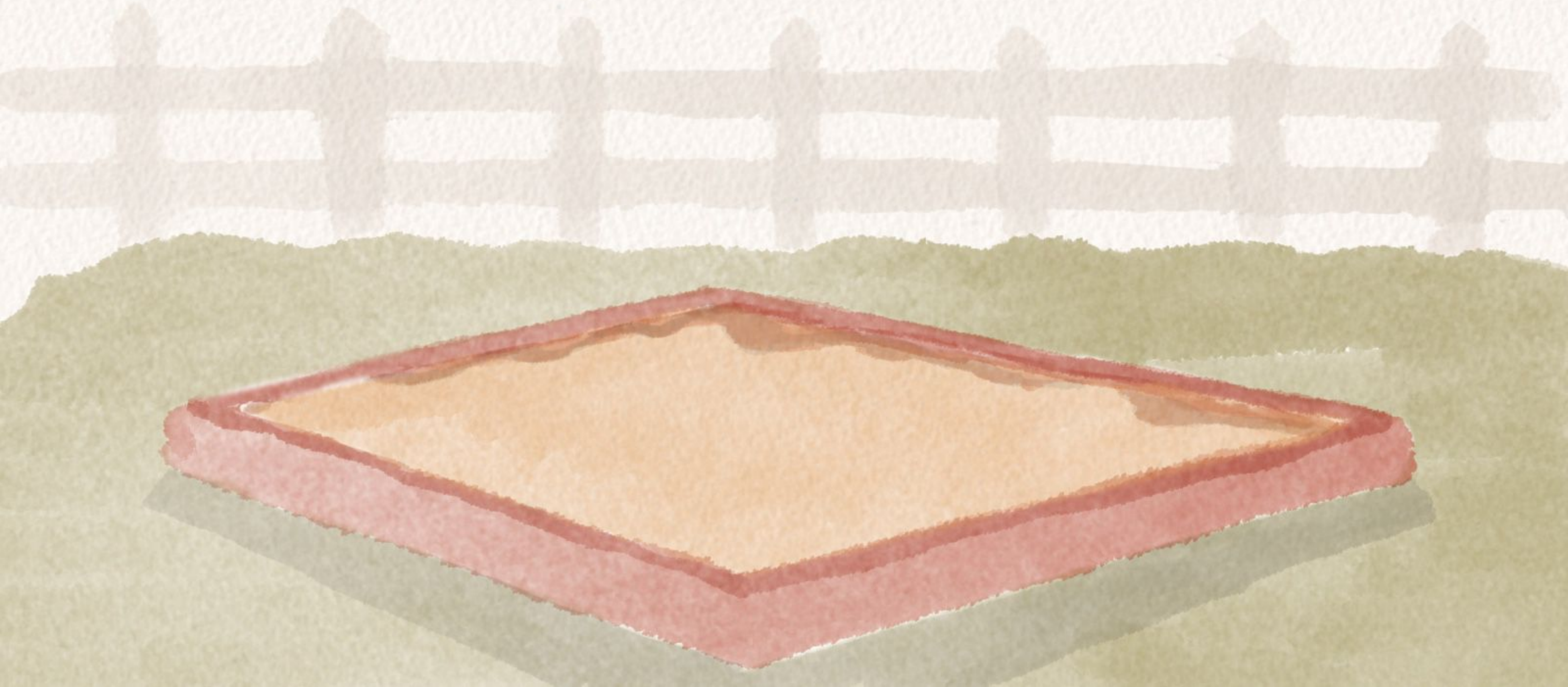
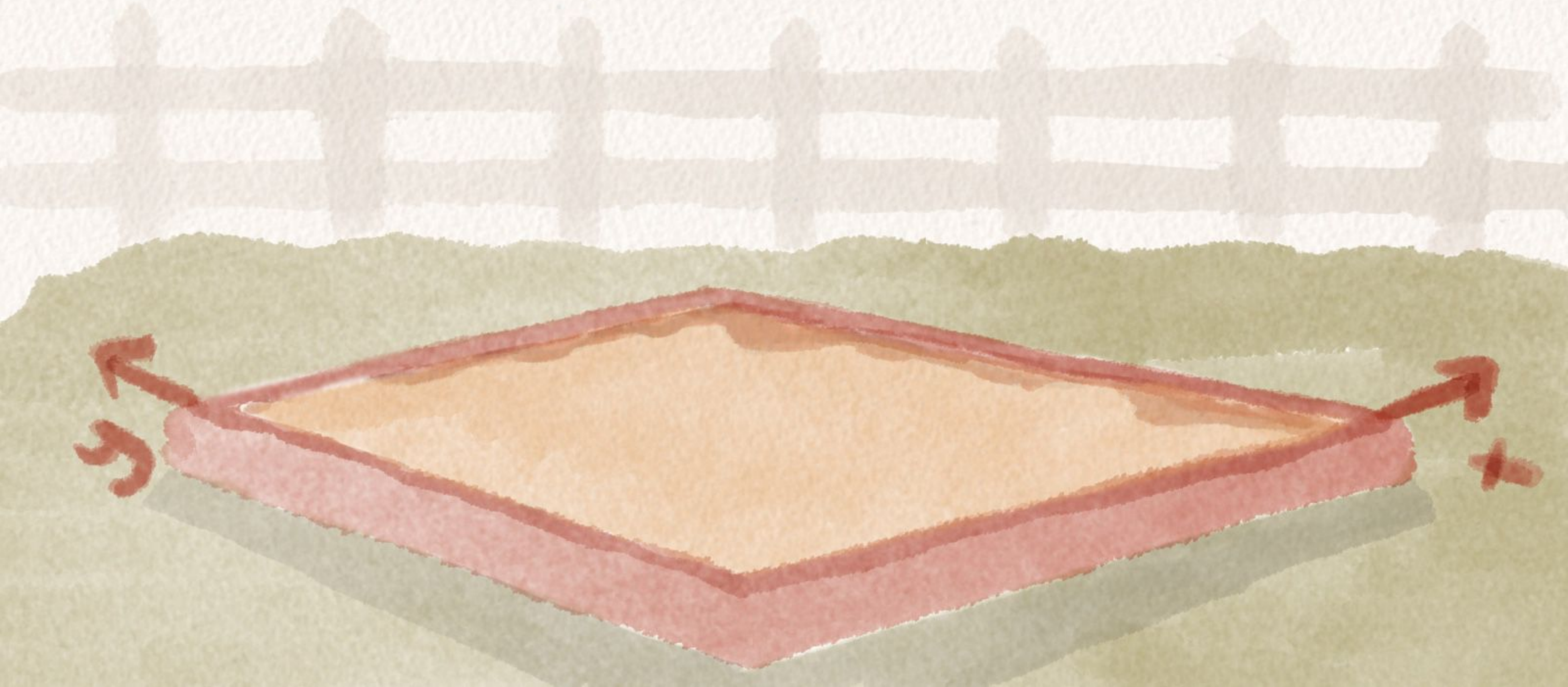
Thm (Combinatorial Nullstellensatz)



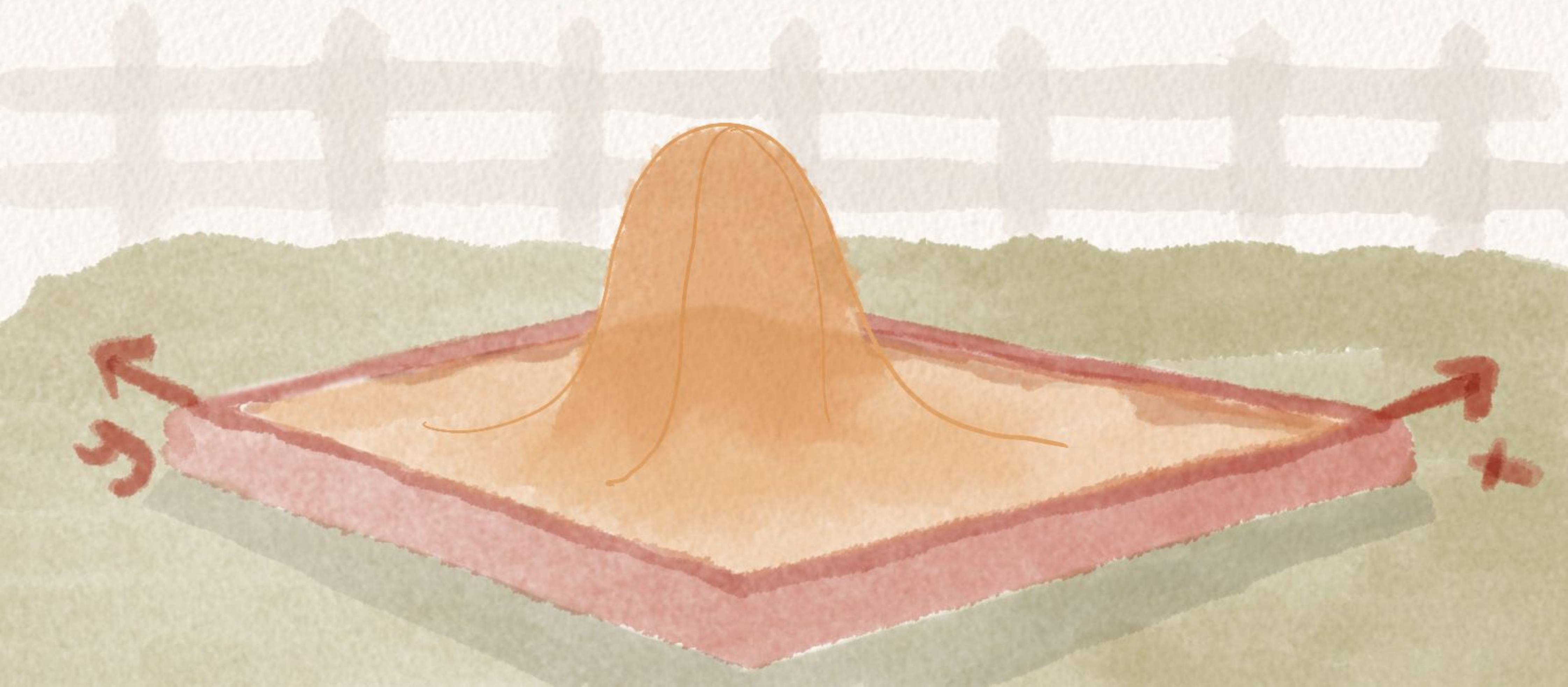
Thm (Combinatorial Nullstellensatz)

Consider a field F



Thm (Combinatorial Nullstellensatz)

Consider a field F containing polynomial $f(x_1, \ldots x_n)$.



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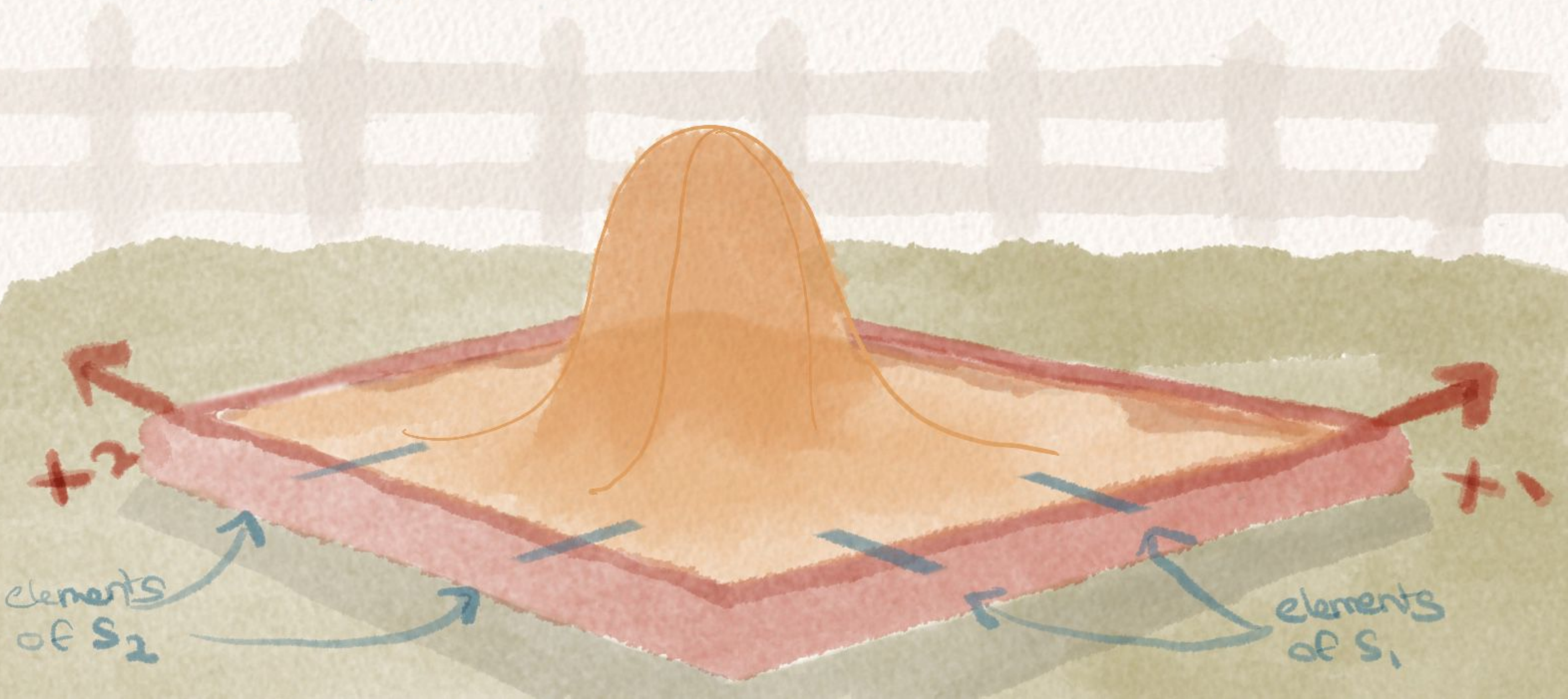
Choose $S_1 \ldots S_n$ as subsets of F.



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Consider a field F containing polynomial $f(x_1 \ldots x_n)$.

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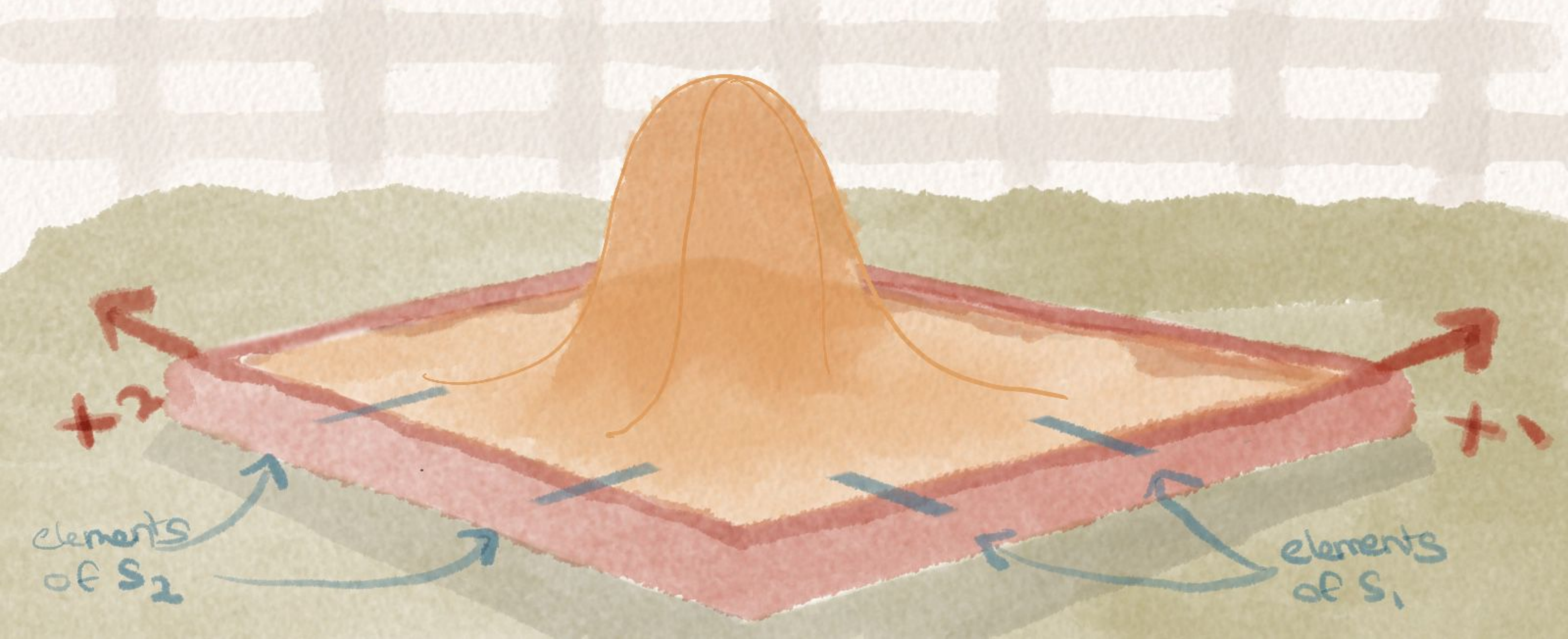

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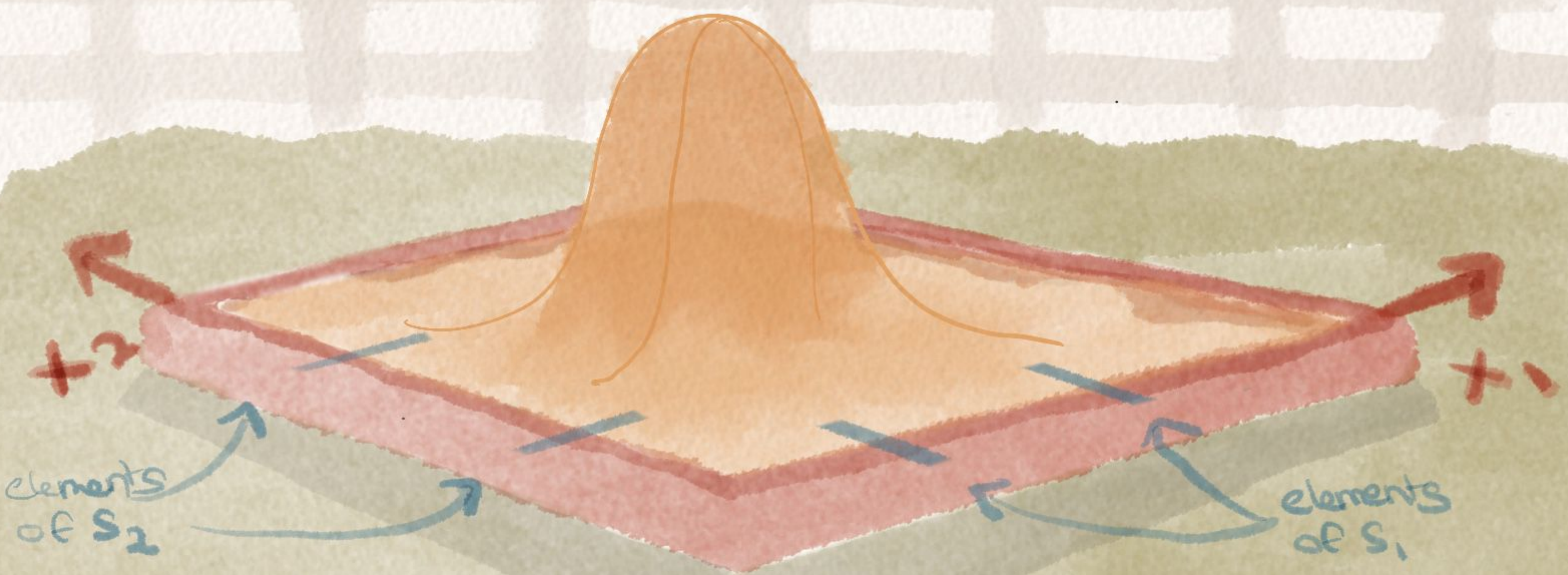
If $|S_i| > t_i$ for all i



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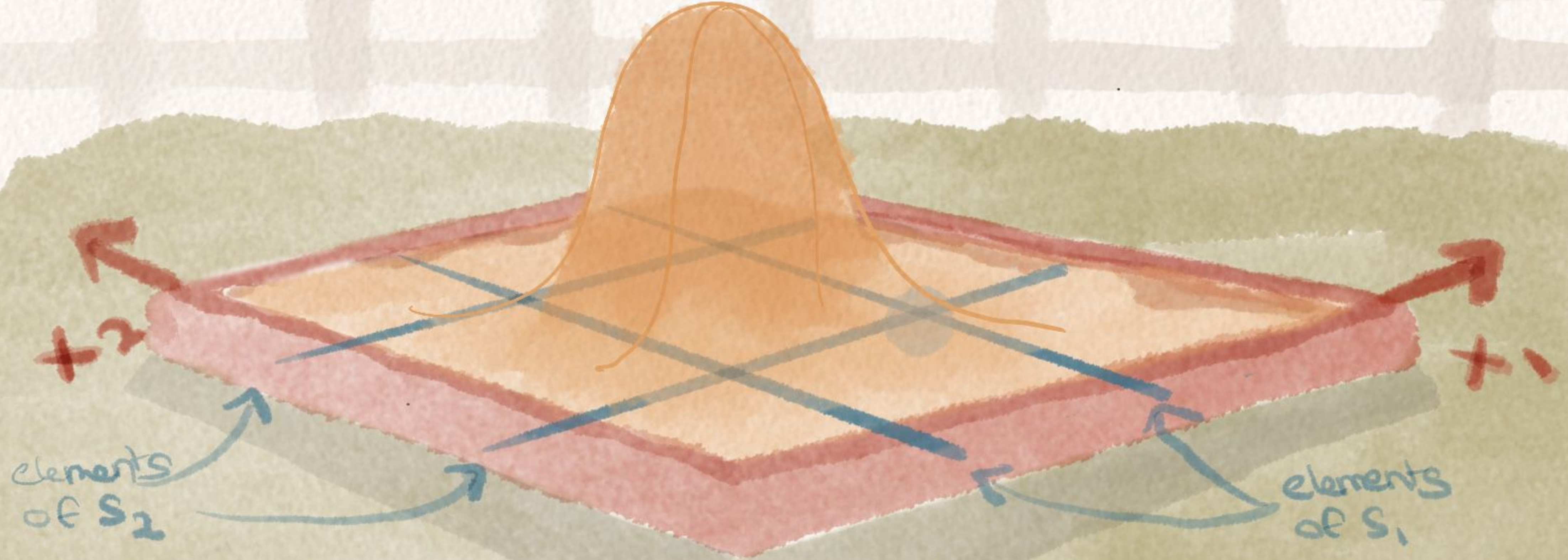
If $|S_i| > t_i$ for all i, then $\exists\, s_1 \in S_1 \dots s_n \in S_n$ s.t. $f(s_1 \dots s_n) \neq 0$.



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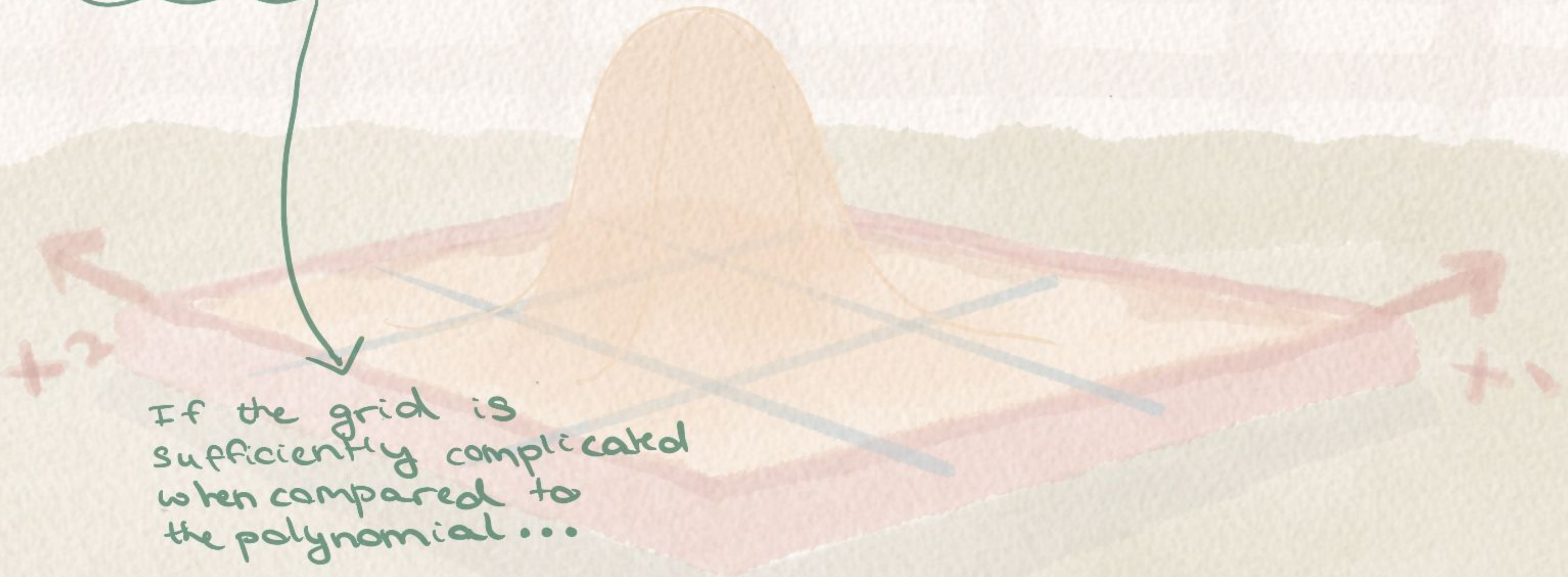

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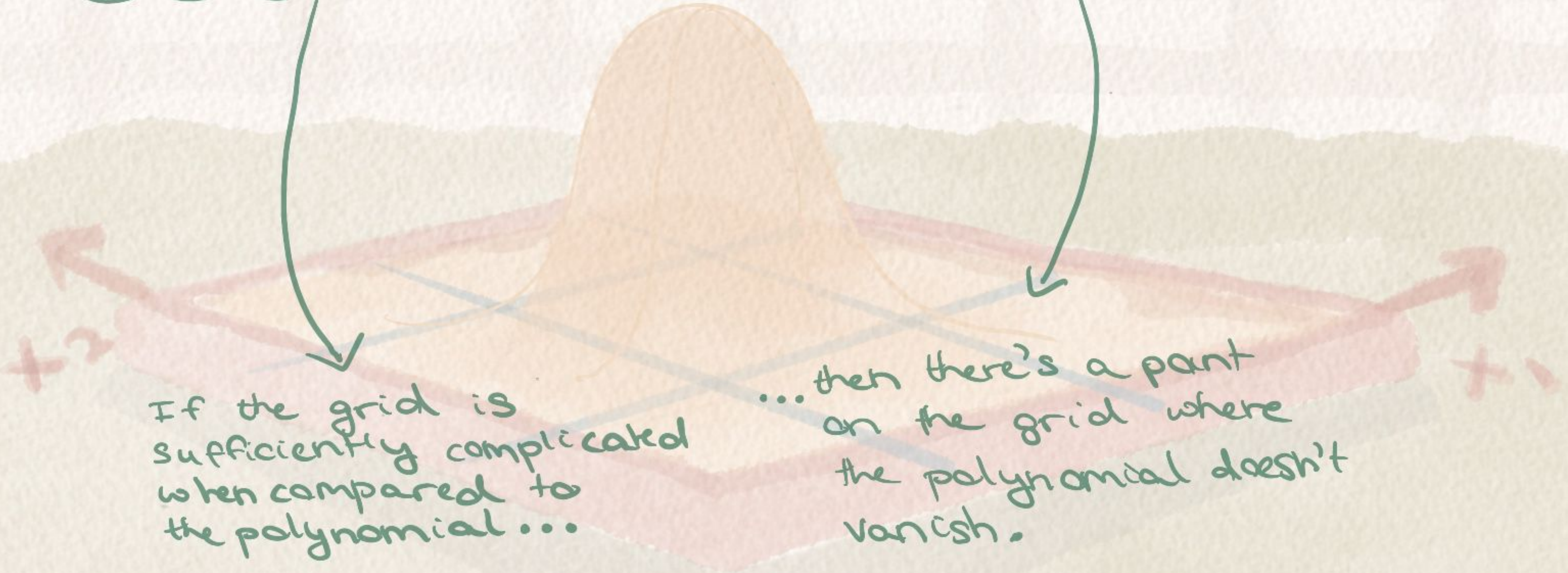

Theorem (Matching) $\rightsquigarrow$ Proof (Comb. Null.)

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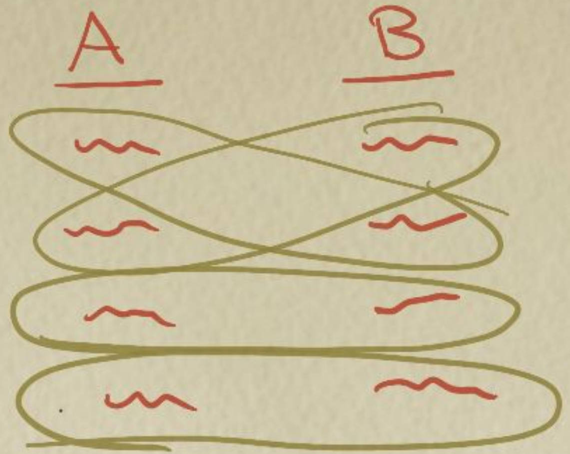

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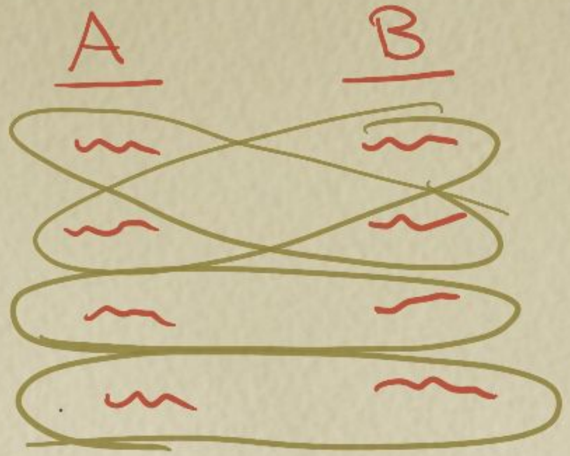
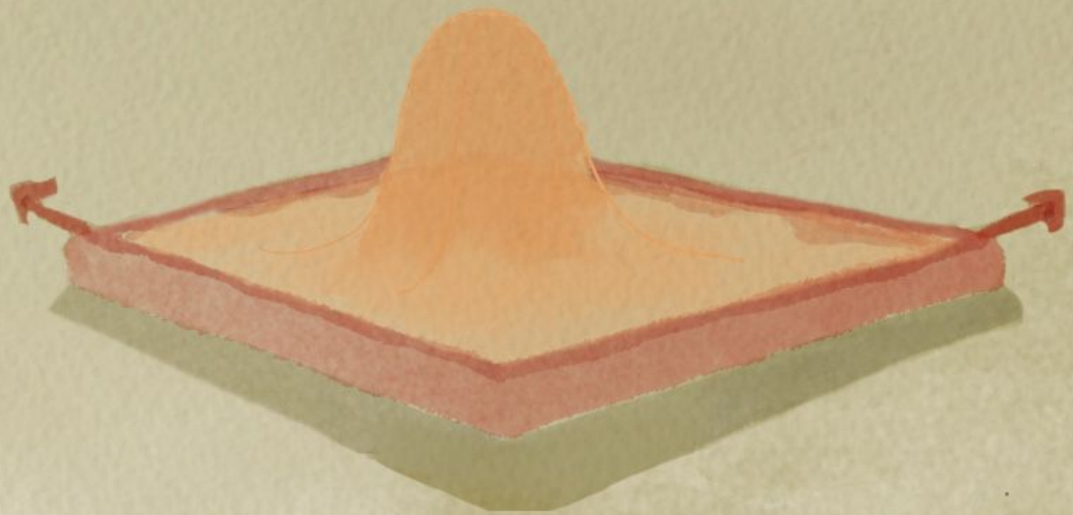
Consider the polynomial in $\mathbb{Z}_p$:

$$f(x_1, \dots x_k) = \prod_{1 \le i < j \le k} (x_i - x_j) \prod_{1 \le i < j \le k} (a_i + x_i - a_j - x_j)$$





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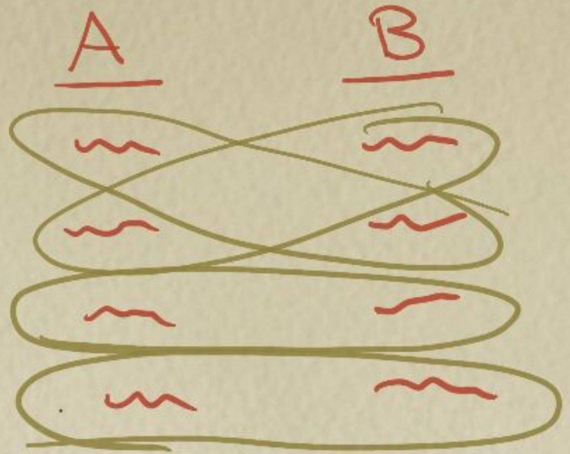
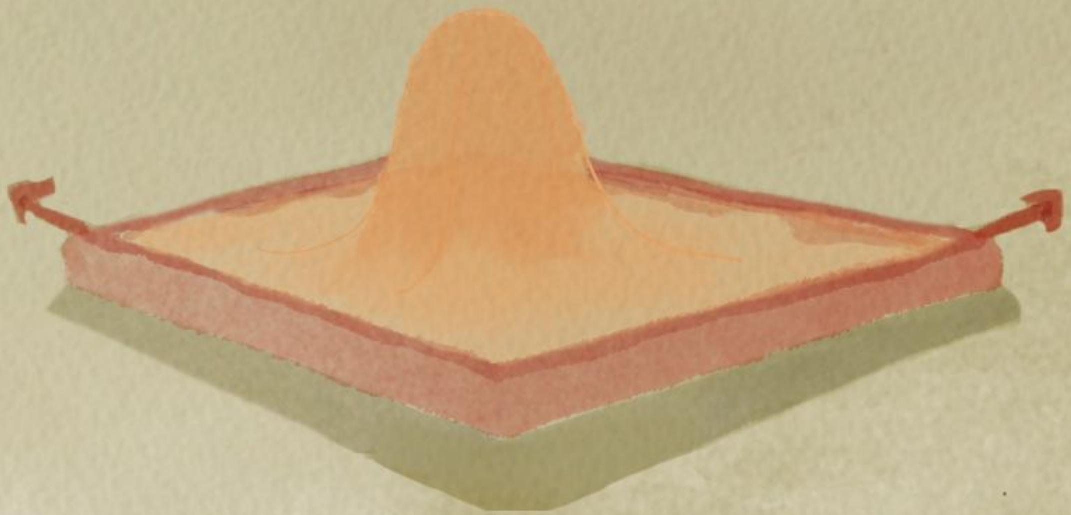
Note the highest degree monomial has form $\prod_{i=1}^{k} x_i^{k-1}$, and non-zero coeff.





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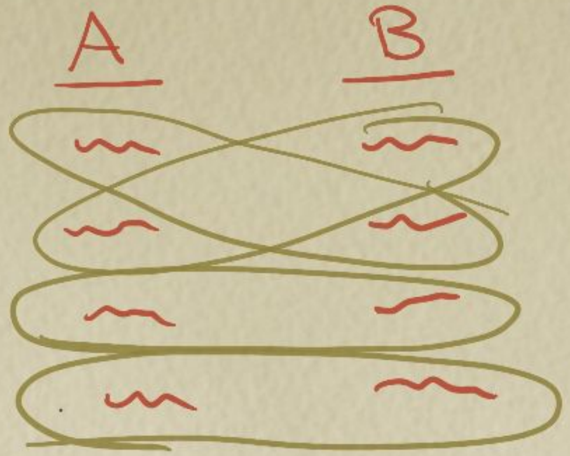

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By Vandermonde identity, the coefficient is $(-1)^{\binom{k}{2}} k!$, which is nonzero mod p because $k < p$.

Theorem (Matching)

Consider a multiset A and set B, each of cardinality $k < p$, where p is prime.

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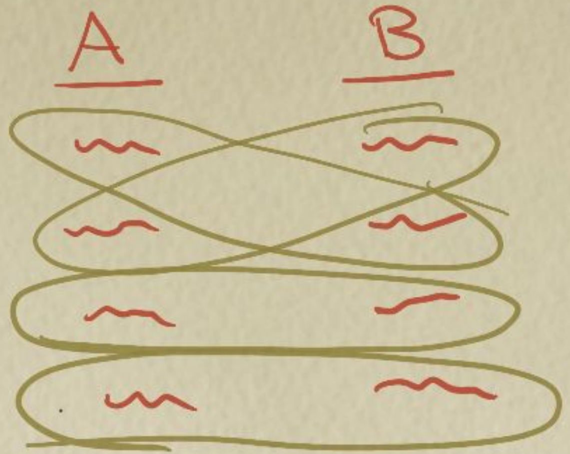

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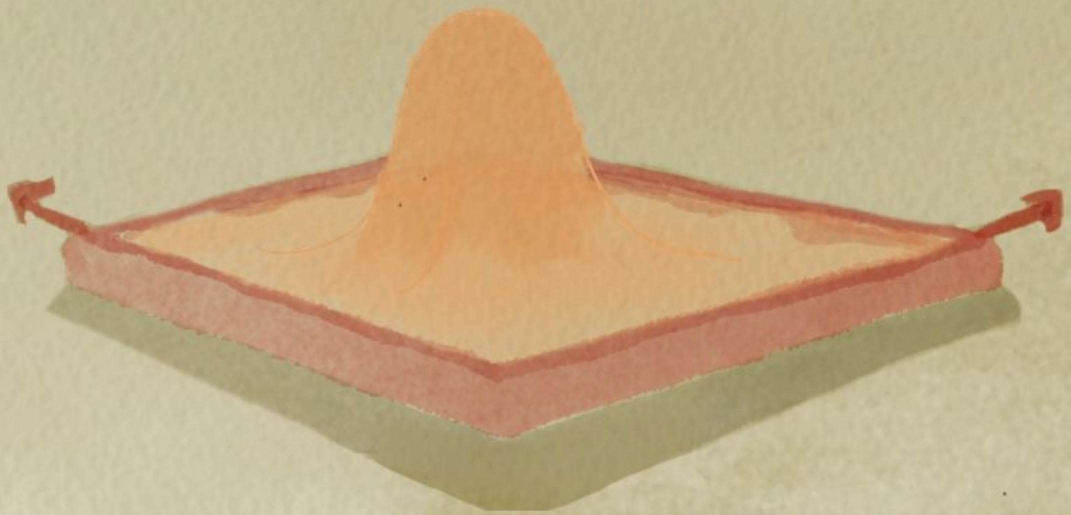
So $t_1, \dots t_k = k-1$.



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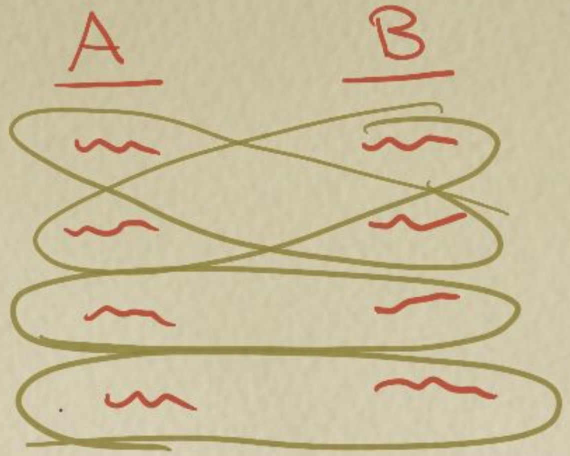
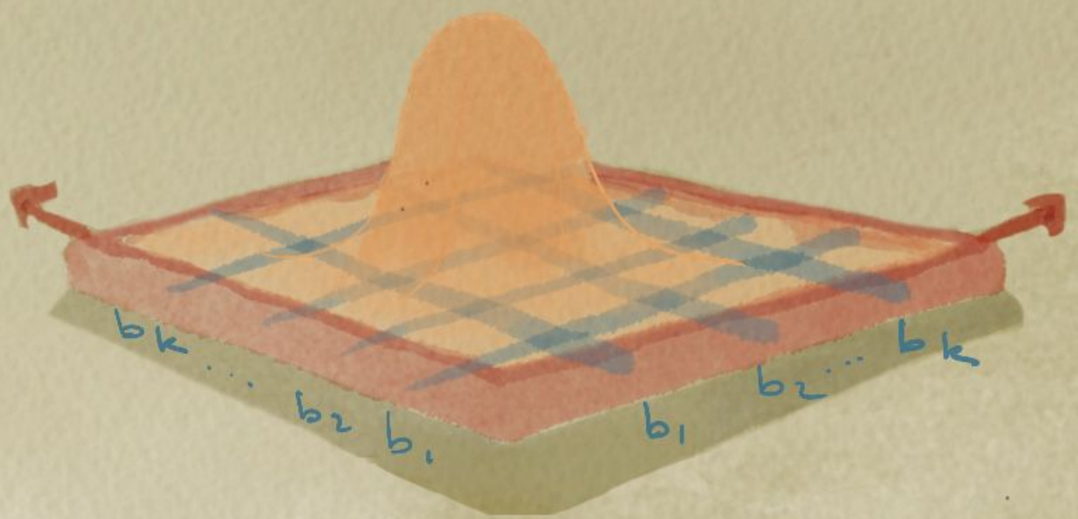
So $t_1, \dots t_k = k-1$. Choose $S_1, \dots S_k = B$.

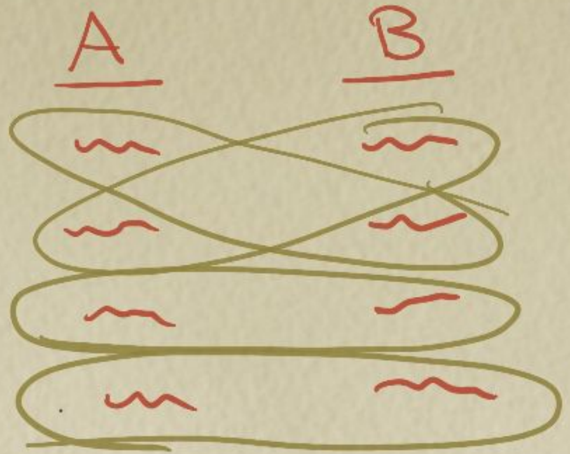




Theorem (Matching) $\rightsquigarrow$ Proof (Comb. Null.)

Consider a multiset A and set B, each of cardinality $k < p$, where p is prime.

Then there exists a numbering of elements $a_1 \dots a_k$ and $b_1 \dots b_k$ s.t. $a_i + b_i \pmod p$ are unique.



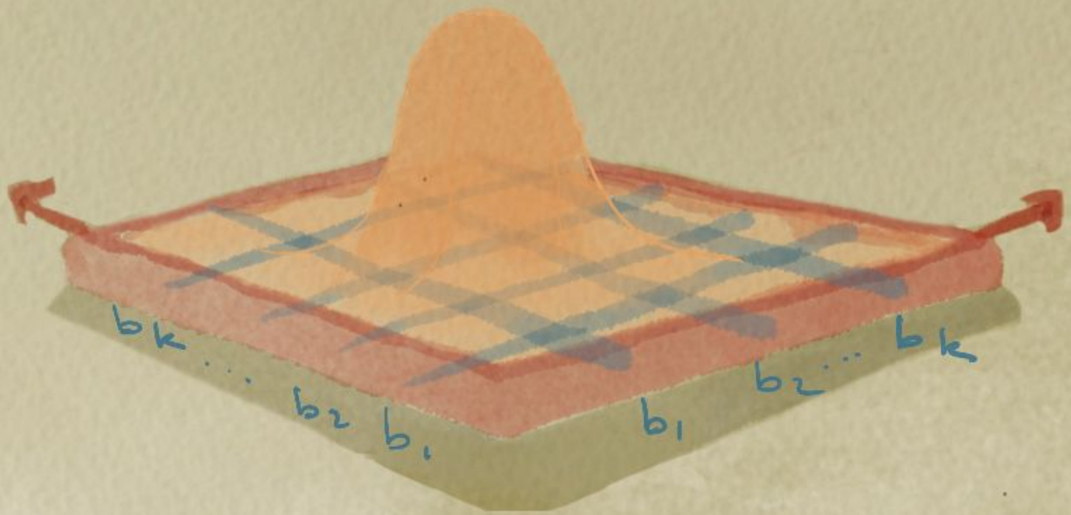
Consider the polynomial in $\mathbb{Z}_p$:

$$f(x_1, \dots x_k) = \prod_{1 \le i < j \le k} (x_i - x_j) \prod_{1 \le i < j \le k} (a_i + x_i - a_j - x_j)$$

Note the highest degree monomial has form $\prod_{i=1}^{k} x_i^{k-1}$, and non-zero coeff.

So $t_1, \dots t_k = k-1$. Choose $S_1, \dots S_k = B$.

Then $|S_i| > t_i$.



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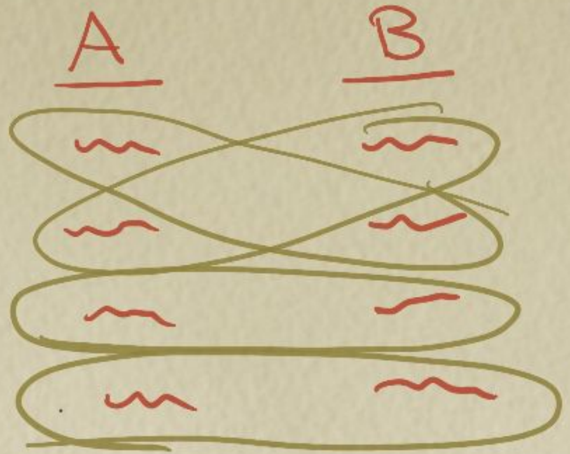

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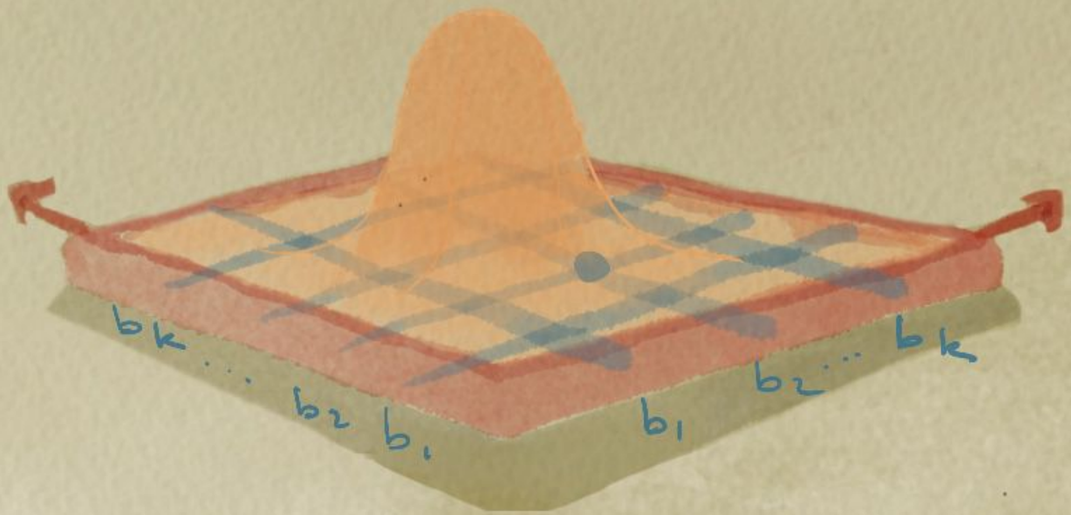
Then $|S_i| > t_i$.

So $\exists (b_1 \dots b_k)$ s.t. $\prod_{1 \le i < j \le k} (a_i + b_i - a_j - b_j) \neq 0$.



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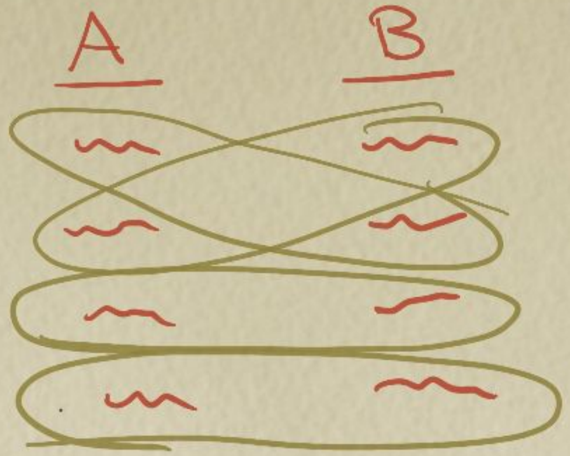

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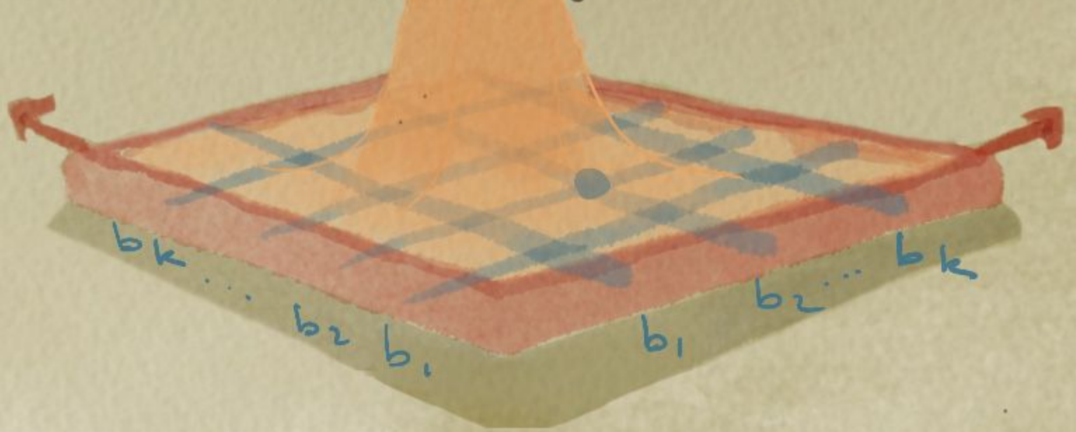
Thus, the sums $a_i + b_i$ are unique for some numbering of a_i and b_i.



Additive Latin Transversals

using Combinatorial Nullstellensatz

- → Discuss Latin transversals conjecture
- → Propose variant of Latin transversals conjecture
- → Reformulate as matching problem
- → Reformulate as Combinatorial Nullstellensatz problem